Summary: This work derives possible statistical options of crater equilibrium by focusing on their cumulative size-frequency distribution (CSFD) slope powers. One of the findings is that the crater equilibrium CSFD slope power shallower than $-2$ can result from both shallow and steep crater production CSFD slopes if small craters can effectively erase large ones, mainly caused by topographic diffusion.

Introduction: Terrestrial planets have been exposed to severe impact bombardments over their lifetimes so that their surfaces exhibit heavily cratered conditions. Some have experienced so many crater emplacements that existing craters are erased by new ones. This crater population evolution process (i.e., crater erasure and creation) has challenged constraining the impact histories because crater erasure has been considered to eliminate historical surface records. The question is whether the observed crater population still gives clues on crater production histories.

Issues on crater equilibrium: The critical unresolved mechanism is crater equilibrium, in which the number of observed craters reaches a balance between generation and erasure and so apparently remains unchanged over time [1,2]. Crater equilibrium uniquely features the observed crater population. Consider two types of CSFDs: the produced crater CSFD, $C_t = \Phi D^{-\eta}$, where $D$ is the existing crater diameter, and $\Phi$ and $\eta (-\eta$ is the slope power) are scalar parameters, and the crater equilibrium CSFD, $C_c|_{eqm} = \Psi D^{-\lambda}$, where $\Psi$ and $\lambda$ are scalar parameters. $C_c$ can include primary and secondary craters in a local area, which play a critical role in crater equilibrium [3,4]. For an ideal case, $\eta > 2$ (i.e., a steep crater production CSFD) yields $\lambda = 2$, which has widely been reviewed. On the other hand, $\eta < 2$ (i.e., a shallow crater production CSFD) leads to $\lambda = \eta$, which may contribute to the large crater population on the Moon [e.g., 5, 6].

Substantial efforts have been made to detail crater equilibrium in many ways [e.g., 6-10]. Xiao and Werner [7] reported that the surface condition might be a significant contributor to crater equilibrium, meaning that crater equilibrium should have multiple patterns (particularly for smaller craters). Minton et al. [9] reported that the ideal $\beta = 2$ case stems from a special condition that the degradation states (in the sense of diffusion processes) only depend on $D^2$, which uniquely makes these coefficients nondimensional. This finding is consistent with Hirabayashi et al. [10].

Despite the recent breakthroughs, there are still limited ways to quantitatively identify whether a target surface has reached crater equilibrium. One of the reasons is a lack of statistical understanding of this mechanism. This issue becomes significant when studies explore crater conditions on multiple terrestrial planets, as the crater production and degradation processes are significantly different. The purpose of this work is to formulate possible statistical conditions of the crater population evolution to quantify when and how crater equilibrium occurs.

Mathematical formulations: Our approach applies the analytical framework developed by Hirabayashi [10], who formulated the interactions between newly emplaced craters and pre-existing craters. Consider that the crater equilibrium CSFD is given as

$$\frac{dC_c}{dD}|_{eqm} = \frac{\frac{dC_t}{dD}}{4A D_{\min}^2 k D^2 dD}$$

where $A$ is a surface area, $C_c$ is the counted crater CSFD, $C_t$ is the produced crater CSFD, $D$ is the existing crater diameter, and $\bar{D}$ is the emplaced crater diameter, which is bounded between $D_{\min}$ and $D_{\max}$. The degradation parameter, $k$, defines a ratio of the actual erasure crater number to the crater erasure number for the ideal cookie-cutting case [10]. If $k \geq 1$, one crater emplacement can erase more efficiently than simple two-dimensional overlapping; otherwise, it can only partially degrade existing craters. We write this parameter as:

$$k = \begin{cases} \alpha D^\beta \left( \frac{\bar{D}}{D} \right)^{\gamma_1}, & \text{if } \bar{D} \leq D \\ \alpha D^\beta \left( \frac{\bar{D}}{D} \right)^{\gamma_2}, & \text{if } \bar{D} > D \end{cases}$$

where $\alpha$ and $\beta$ are both scalar parameters, and $\gamma_1$ and $\gamma_2$ are the scaling powers to describe how $k$ changes with $\bar{D}$ relative to $D$.

Figure 1 shows a schematic of this parameter. When $\bar{D} \leq D$, a newly emplaced crater is smaller than an existing crater with degradation, meaning that $k$ is relatively small ($k \leq \alpha D^\beta$) and represents topographic diffusion by smaller crater emplacements. On the other hand, when $\bar{D} > D$, a newly emplaced crater is larger, describing that this parameter is relatively large ($k \geq \alpha D^\beta$) and represents cookie-cutting and ejecta blanketing. The transition diameter between these two regimes, $\bar{D} = D$, results from how a crater emplacement erases the three-dimensional topographic features of existing craters. Minton et al. [9] elegantly described this transition diameter by considering that the degradation state that an existing crater becomes invisible completely equals the degradation state that
one crater emplacement can degrade an existing crater. In this sense, it is necessary to consider how the crater is identified by crater counting (i.e., human or machine counting factors) [9]. However, we do not consider this issue in this study.

**Results:** This report focuses on when the slope powers of the degradation parameter, $\gamma_1$ and $\gamma_2$, are both positive. This means that larger crater emplacements, i.e., $D$, are more capable of degrading smaller existing craters, i.e., $D$. The derivative of $C_e|_{eqm}$ with respect to $D$ is provided as

$$\frac{dC_e|_{eqm}}{dD} = -\frac{4q}{\pi} D^{3-\beta} \left[ \left( \frac{\gamma_1 - \gamma_2}{(\eta - 2 - \gamma_1)(\eta - 2 - \gamma_2)} \right) \left( \frac{D_{min}}{D} \right)^{-\eta+2+\gamma_1} \right] + \frac{1}{\eta - 2 - \gamma_1} \left( \frac{D_{max}}{D} \right)^{-\eta+2+\gamma_2}$$

This equation is always negative to satisfy $C_e|_{eqm}$ is always positive. The derived equation above, however, encounters singularity issues at $\eta - 2 - \gamma_1 = 0$ or $\eta - 2 - \gamma_2 = 0$. The present study does not consider these conditions as this issue came from the non-smooth transition defined for $k$. Integrating this equation with respect to $D$ yields $C_e|_{eqm}$. There are three terms that control the crater equilibrium power slopes, i.e., the terms within the large brackets on the right side. Depending on which term is dominant, $C_e|_{eqm}$’s slope power, $\lambda$, changes: $-2 - \beta$ (first term), $-\eta - \beta + \gamma_1$ (second term), and $-\eta - \beta + \gamma_2$ (last term).

Table 1 shows three cases that could appear depending on $\gamma_1$ and $\gamma_2$. The $\gamma_1 > \gamma_2$ case defines the $k$ value being small at $D/D << 1$ (inefficient topographic diffusion) and remaining around a $D^\beta$ at $D/D \gg 1$ (degradation dominated by cookie cutters). The $\gamma_1 \leq \gamma_2$ cases define the $k$ value giving effective degradation to smaller and larger craters (by topographic diffusion for small craters and ejecta blanketing and others for large craters). The subcases with different sines of $\eta - 2 - \gamma_1$ and $\eta - 2 - \gamma_2$ further find multiple slope power conditions.

The slope powers, $-\eta - \beta + \gamma_1$ and $-\eta - \beta + \gamma_2$, have thresholds of crater diameters. For the term with the $-\eta - \beta + \gamma_1$ slope power, $D_{min}$ should not be infinitesimally small; in other words, $D_{min}$ is finite so that this term is bounded. Similarly, with the $-\eta - \beta + \gamma_2$ slope power, $D_{max}$ should not be infinitely large to limit the term. The consequence of setting such ill-defined conditions leads to $C_e|_{eqm} = 0$. The major reason is that crater degradation for these cases is so strong that taking limits, $D_{min} \to 0$ or $D_{max} \to \infty$, wipes out all existing craters.

**Discussions:** This study provides seven subcases under three cases. For example, the subcases under the $\gamma_1 > \gamma_2$ case has widely been considered in the literature (Figure 1a). The first subcase, $\eta - 2 - \gamma_1 < 0$ and $\eta - 2 - \gamma_2 < 0$, represents a shallower crater production. When ejecta blanketing is negligible, and cookie-cutter is a primary contributor to crater erasure, $\gamma_2 \sim 0$, leading to $\eta < 2$. Without considering the size dependence of the transition, i.e., $\beta = 0$, the equilibrium slope power becomes $-\eta (\lambda = \eta)$. The second subcase is the most common. This case leads to $\eta > 2$, given $\eta - 2 - \gamma_2 > 0$. The resulting slope power is $-2 - \beta$; if there is no size dependence term, it becomes $-2 (\lambda = 2)$. In this case, how steep the production CSFD does not matter for the equilibrium CSFD slope. The last subcase may cause a steeper crater production CSFD to produce an equilibrium CSFD slope shallower than $-2$, which is $-\eta - \beta + \gamma_1$. Because $\eta - 2 - \gamma_1 > 0$, the crater production CSFD slope power should be steeper than $-2 - \gamma_1$. If $\beta = 0$, $\gamma_1 > 0$ leads to $-\eta + \gamma_1 - \eta$. This subcase results from relatively higher topographic diffusion degradation, inferring that it should be cautious to access shallow equilibrium CSFD slopes.

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![Figure 1. Degradation parameter, k, as a function of D/D. Each case shows how k behaves at given D/D.](image)

<table>
<thead>
<tr>
<th>$\gamma_1$ vs $\gamma_2$</th>
<th>$\eta - 2 - \gamma_1$</th>
<th>$\eta - 2 - \gamma_2$</th>
<th>Slope power, $\lambda$</th>
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<td>$\gamma_1 &gt; \gamma_2$</td>
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