An Analytic Model of Stable and Unstable Orbital Resonance. V. Jayam[1,†], K. Thaman[2], J. Kim[3], S. Jain[4], Q. Xue[5], A. Dutta[6], [1]The Wheatley School, USA (virajjayam@gmail.com), [2]Eicher School, India, [3]Monta Vista High School, USA, [4]Evergreen Valley High School, USA, [5]John Fraser Secondary School, Canada, [6]Wayzata High School, USA.

Introduction: The goal of this project is to develop a model for orbital resonance, both in the case when it leads to chaotic orbits or to stable configurations. The study of resonance is of great interest in astronomy as it can give clues to the formation of the early solar system by analyzing where certain objects are, and where there is a characteristic lack of objects. Through techniques from Newtonian mechanics, this paper predicts the relative strengths of Kirkwood gaps in the main asteroid belt. Additionally, it builds off of prior work to build a new and more complete approximation of the liberation period in stable resonant systems such as Saturn's moons, Titan and Hyperion.

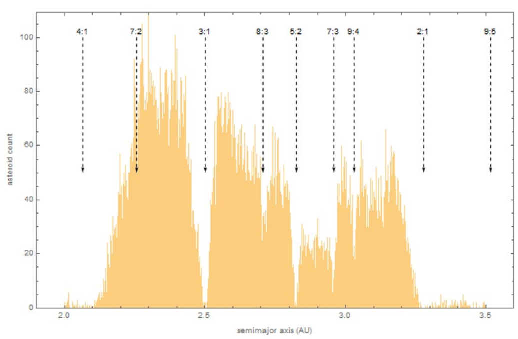

Figure 1: A histogram of Kirkwood Gaps and the locations of orbital resonances, plotted with Mathematica

Methodology:

Elementary Model: First, we adapt a simplistic model that can explain the relative strengths of the Kirkwood Gaps. Consider an asteroid orbiting around Jupiter (Figure 3). Let us assume that the two objects begin in conjunction. We base our model off the following assumption: If we superimpose the larger object at all the locations it can be in when the smaller object makes one full revolution starting from conjunction, then the strength of orbital resonance directly depends on the average of all the superimposed forces.

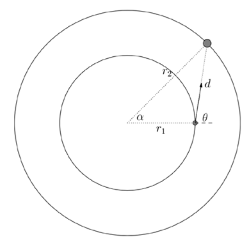

Figure 2: The inner mass, on average, experiences an outwards radial force due to the outer mass.

The superimposed strength of the resonance is calculated then to be (depicted $F_{r,\mathrm{avg}} \propto \frac{1}{m}\sum_{k=1}^{m} \frac{\cos\frac{2\pi k}{m} - f^{2/3}}{\left(f^{4/3}+1-2f^{2/3}\cos\frac{2\pi k}{m}\right)^{3/2}}$ in Figure 2)

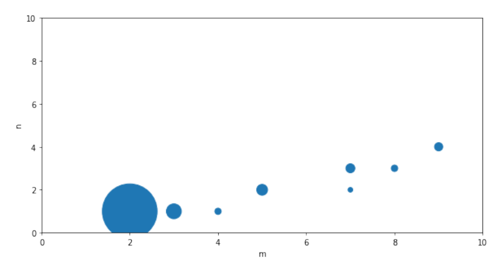

FIG. 3. Average force at all possible resonant frequencies with m, n < 10. The size of the bubble represents the relative strengths. Frequencies that are not relatively prime, or exist outside of the main belt are not included.

Stable Resonances: This periodic amplification can wreck havoc to objects in a perfectly circular orbit, but it can lead to stable resonance as well in certain cases

where the orbits have a nonzero eccentricity. This is common between different planets and moons.
A famous example of stable resonances is found in the moons of Saturn, or to be more specific, Hyperion and Titan. Hyperion gains an uncertainty of 0.123ε in its orbit due to the alluding orbital resonance caused from Titan. In turn, we can see a near perfect circular orbit produced from Titan, and an elliptical orbit created by Hyperion. If the conjunction point is located when moving from pericenter to apocenter, Hyperion receives an out- ward velocity component while the radial component of Titans gravitational attraction pulls Hyperion inwards. Because of this, Hyperion's energy and angular momentum decrease which, in turn, also decreases the semi- major axis and period of Hyperion. This effect is also enhanced because the point where the pull is the strongest is right before the conjunction point, while Titan, which is faster, trails right behind Hyperion. As a consequence, Hyperion speeds up, which makes the conjunction move towards the apocenter of the orbit. Another case exists if the conjunction exists when Hyperion moves from apocenter to pericenter. In this scenario, Titan's pull in- creases the gravitational energy and angular momentum, slowing the moon down. However, this effect still moves the conjunction to apocenter.

Following the techniques outlined by Agol et al. 2005, we found a closed form expression for the period of liberations discussed above. However, they provided only the general proportionality between liberation period, eccentricity, and the resonant frequency. We took a step forward and investigate the general proportionality factor, and how it depends on the semi-major axis and if and how it depends on the amplitude of oscillations. We obtain that the period of liberation as 1.68 years, surprisingly close to the actual liberation period of 1.75 years!

Results: In this paper, we have analyzed and created a mathematical model for two cases of orbital resonance in the Solar System: Kirkwood Gaps in the Asteroid Belt, and Saturn's two moons Hyperion and Titan. The former was modeled as two concentric and circular orbits with the outer mass, Jupiter, much larger than the inner mass, an asteroid. By taking the average radial component of force that Jupiter exerts on the asteroid at regular time intervals, we found that, at certain resonant frequencies, the average force is much stronger than at others. To a degree of remarkable accuracy, these points corresponded with the points at which the Kirkwood Gaps occur.

Secondly, the Saturn-Titan-Hyperion system was modeled as a large mass, Titan, in a circular orbit, and a smaller mass, Hyperion, in a larger and elliptical orbit. During the subsequent 4:3 resonant motion, Hyperion periodically gains and loses energy, causing its orbit to change shape. The time for the system to completely return to its initial state, also known as the liberation period, was successfully approximated to an order of magnitude.

References:

[1] Agol E. et al. (2005) *Monthly Notices of the Royal Astronomical Society*, **359**, [2] Bevilacqua (1980), *The Moon and the Planets*, **22**, [3] Jacobson (2006), *The Astronomical Journal*, **132**, [4] Wolfram-Alpha Knowledgebase, *Minor Planet Data Source Information*