MATHEMATICAL THEORY OF THERMAL INERTIA REVISITED 1: IMPROVING OUR UNDERSTANDING OF MARTIAN THERMOPHYSICAL PROPERTIES THROUGH ANALOGOUS EXAMPLES OF *PERIODIC DIFFUSIVE INERTIAS*.

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Introduction: In Planetary Science, thermal inertia has been defined the ability of a material to resist a temperature change when applying a periodic forcing function. Thermal inertia is used to answer scientific questions about the Martian surface and is used to constrain engineering requirements. However, one not familiar with its derivation, may inquire the following: 1) Why are the factors of thermal inertia under a radical? 2) What is the significance of an inverse roots second? 3) How does thermal inertia affect the heat flow i.e. what is the applicable equation? 4) Are there any analogous scenarios to improve conceptual understanding?

We reproduce the mathematical derivation of thermal inertia from first principles to recall this parameter's origin and application. Finally we provide a set of analogous equations and propose a new term, *periodic diffusive inertia* that describes the ability of a potential variable (e.g. temperature) to change within a class of scenarios described by a diffusion equation undergoing a periodic boundary condition.

Background: Thermal inertia is a thermophysical property of a material and is defined as the square root of the thermal conductivity, specific heat, and density—with the units of joules per square meter per kelvin per root second.

$$I \equiv \sqrt{k\rho c} \quad [\mathrm{J \cdot m^{-2} \cdot K^{-1} \cdot s^{-1/2}}] \qquad \textbf{Eq. 1}$$

The thermal inertia of a surface is calculated from diurnal temperature changes from fly-by/orbiting spacecraft at Mars for ~50 years e.g. IR 6/7, IRIS, IRTM, TES, THEMIS, and OMEGA [1-6]. Detailed models calculate and map thermal inertia from observed orbital temperatures [4,7,8], and laboratory investigations have furthered its implications e.g. [9,10]. *In-situ* thermal inertia measurements have been collected with Mini-TES [11], and REMS is collecting thermal inertia *in situ* at Gale Crater [12]. The THEMIS team has recently published quantitative thermal inertia maps [13]. The martian literature is rich with thermal inertia as a scientific proxy to study particle size, outcrops of bedrock, induration, volatile/ice content, and more. Knowledge of the surface material has many engineering implications such as landing site selection and rover traverse planning [14,15].

Derivation: We consult the works of Solid Heat Conduction from *Ingersoll, Schneider, Carslaw and Jeager,* and *Wesselink* for the derivation [16-19]. The problem consists of a homogenous, semi-infinite solid excited by a steady, periodic forcing at the free surface. Via the methodology shown in figure 1, we reproduce a function for the instantaneous heat (Eq. 2). We see that 1) the square root for thermal inertia arises from the solution to the diffusion equation, a parabolic partial differential equation; 2) the inverse-root second, when multiplied by the root of the frequency, is "unrooted"; this is also the case for the e-folding skin depth; and 3) the thermal inertia is a coefficient of the equation for Periodic Thermal Flux, with some rearrangement, this is in agreement with flux equations in contemporary texts [19,20].

$$\Phi_q = \theta_{0_1}\sqrt{k\rho c}\sqrt{\omega}\cos\left(\omega t + \frac{\pi}{4}\right) \ [\mathrm{J \cdot m^{-2} \cdot s^{-1}}] \qquad \textbf{Eq. 2}$$

Analogies: Analogies are often applied to physical concepts to understand it from a different vantage point e.g. the mass-spring-damper mechanical system and the inductor-capacitor-resistor circuit. Circuits were used to simulate diffusion problems of heat and groundwater flow [21,22]. To answer question 4, we show that *thermal inertia* has analogies to the diffusion of molecular mass, the momentum of a fluid between oscillating plates, the hydraulic head in an aquifer undergoing periodic charging, and the alternating current in an electrical circuit.

For a steady state case, these scenarios are solely defined by a transport law that describes the change in some quantity of potential with respect to distance. These equations all contain a *conductivity coefficient*.

For the transient case, a corresponding equation exists for each of these scenarios that describes the storage of each principal quantity with respect to time. Each of these equations contains a *capacity coefficient*.

Periodic Diffusive Inertia: Here we have found a general equation that contains the amplitude of the potential term, the square root of the conductivity and capacity coefficients, the square root of the frequency, and a cosine term that leads the boundary condition by π/4. This is the general periodic flux equation (Eq. 3). **Furthermore it can be recognized that in each of these scenarios an analogous thermal inertia term is equal to the square root of the conductivity term multiplied by the capacity term.** One could argue that the frequency should be included here—to remedy the ambiguous inverse root second—however the frequency is not associated with the physical properties of the system, and it has historically been excluded.

In addition, **we propose that this general term be called the *periodic diffusive inertia* (ς)** (Eq. 4)**.** Unlike linear inertia i.e. *mass* or rotational inertia i.e. *moment of inertia, periodic diffusive inertia* is used specifically for a system described by the diffusion equation undergoing a periodic boundary condition.

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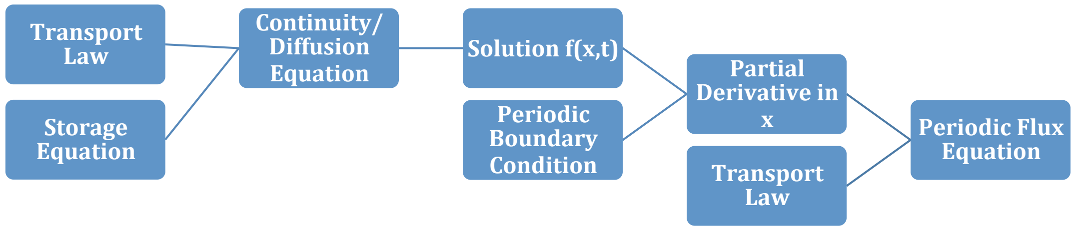

Fig 1. Methodology for deriving the Periodic Flux Equation. The transport law and storage equation form a continuity equation in the form of the diffusion equation. It is solved for a solution f(x,t). The periodic boundary condition refines the equation as a boundary condition. The partial derivative is taken with respect to x. The result is substituted back into the transport law to obtain the final periodic flux equation.

$$\Phi_{Principal\ Flux_{x=0}} = \psi_{Potential\ Amplitude_{x=0}} \cdot \varsigma_{Oscillating\ Diffusive\ Inertia} \cdot \sqrt{\omega} \cdot \cos\left(\omega t + \frac{\pi}{4}\right) \quad \textbf{Eq. 3}$$

$$\varsigma_{Oscillating\ Diffusive\ Inertia} = \sqrt{K_{Conductivity} \cdot C_{Capacity}} \quad \textbf{Eq. 4}$$

Table 1. Analogies

Term	*Thermal Energy*	*Molecular Mass*	*Fluid Momentum*	*Hydraulic Discharge*	*AC Electrical Circuit*
Principal Quantity	Q [J]	Mass M [mol]	Momentum P [N· s]	Discharge Q [m^3]	Charge Q [C]
Flow	q [J· s^{-1}]	Mass Flow $\dot{M}$ [mol·s^{-1}]	Force F [N]	Discharge Rate q [m^3·s^{-1}]	Current I [C·s^{-1}=A]
Principal Flux	Heat Flux Φ [J·m^{-2}·s^{-1}]	Diffusion Flux J [mol·m^{-2}·s^{-1}]	Shear Stress τ [Pa = N·m^{-2}]	Discharge Flux Φ [m·s^{-1}]	Current Density j [A·m^{-2}]
Potential	Temperature θ [K]	Concentration ϕ [mol·m^3]	Velocity v [m·s^{-1}]	Head h [m]	Voltage V [V]
Transport Law	Φ = -k∇T *Fourier's Law*	J=-D∇ϕ *Fick's Law*	τ = -μ∇u *Newton's Law of Viscosity*	Φ = -k ∇h *Darcy's Law*	j=-σ∇V *Ohm's Law*
Conductivity Coefficient	Thermal Conductivity k [J· m^{-1}·K^{-1}·s^{-1}]	Diffusion D [m^2·s^{-1}]	Viscosity μ [Pa· s]	Hydraulic Conductivity K=κ/μ [m· s^{-1}]	Electrical Conductivity σ=l/A·R [Ω^{-1}·m^{-1}]
Storage Equation	$dQ = c\rho \cdot v \frac{\partial\theta}{\partial t} dt$ *Black's Eq.**	$dM = v \frac{\partial\phi}{\partial t} dt$ *Mass Flux*	$dP = \rho \cdot v \frac{\partial v}{\partial t} dt$ *Newton's 2nd Law*	$dQ = S\_m \cdot v \frac{\partial h}{\partial t} dt$ *Groundwater Flow Eq.*	$dq = C \frac{\partial V}{\partial t} dt$ *Capacitance Eq.*
Capacity Coefficient	Volumetric Heat Capacity ρ· c [J·K^{-1}·m^{-3}]	1	Density ρ [kg·m^{-3}]	Specific Storage S_m [m^{-1}]	Capacitance C [s·Ω^{-1}]
Diffusivity Coefficient	$\alpha = \frac{k}{\rho c}$ *Thermal Diffusivity*	D *Diffusivity*	$\upsilon = \frac{\mu}{\rho}$ *Kinematic Viscosity*	$a = \frac{K}{S\_m}$ *Hydraulic Diffusivity*	$a_{LRC} = \frac{1}{R_l C_l} = \frac{l^2}{RC} = \omega_{RC} * l^2$ *Electric Diffusivity*/ Cut-off Frequency*
Diffusion Equation	$\frac{\partial^2\theta_x}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$ *Heat Diffusion*	$\frac{\partial^2\phi_x}{\partial x^2} = \frac{1}{D}\frac{\partial\phi}{\partial t}$ *Fick's 2nd Law*	$\frac{\partial^2 v_x}{\partial y^2} = \frac{1}{\upsilon}\frac{\partial v}{\partial t}$ *Simplified Navier-Stokes Eq.**	$\frac{\partial^2 h}{\partial y^2} = \frac{1}{a}\frac{\partial h}{\partial t}$ *Hydraulic Diffusion**	$\frac{\partial^2 V}{\partial x^2} = R_l C_l \frac{\partial V}{\partial t}$ *Simplified Telegraph Eq. **
Periodic Flux Equation	$\phi_q = \theta_{0_1}\sqrt{k\rho c}\sqrt{\omega} \cdot \cos\left(\omega t + \frac{\pi}{4}\right)$	$J = \phi_{0_1}\sqrt{D}\sqrt{\omega} \cdot \cos\left(\omega t + \frac{\pi}{4}\right)$	$\tau = u_{0_1}\sqrt{\mu \cdot \rho}\sqrt{\omega} \cdot \cos\left(\omega t + \frac{\pi}{4}\right)$	$\Phi = h_{0_1}\sqrt{K \cdot S_m}\sqrt{\omega} \cdot \cos\left(\omega t + \frac{\pi}{4}\right)$	$J = V_{0_1}\sqrt{\sigma \cdot \left(\frac{\epsilon}{d^2}\right)}\sqrt{\omega} \cdot \cos\left(\omega t + \frac{\pi}{4}\right)$
Periodic Diffusive Inertia	$\boldsymbol{\sqrt{k \cdot \rho c}}$ **[J·m^{-2}·K^{-1}·$s^{-1/2}$]** ***Thermal Inertia***	$\boldsymbol{\sqrt{D}}$ **[m·$s^{-1/2}$]** ***Diffusive Inertia****	$\boldsymbol{\sqrt{\mu \cdot \rho}}$ **[kg·m^{-2}·$s^{-1/2}$]** ***Fluid Inertia****	$\boldsymbol{\sqrt{K \cdot S_m}}$ **[$s^{-1/2}$]** ***Hydraulic Inertia****	$\boldsymbol{\frac{\sqrt{\sigma \cdot C}}{l^2}}$ **[$s^{1/2}$·Ω^{-1}·m^{-2}]** ***Electrical Inertia****

*Placeholder names not found in literature. **This likely applies to other scenarios e.g. quantum, biological, and statistics