Limitations of the cube method for assessing large constellations

Hugh G. Lewis[(1)], Samuel Diserens[(1)], Timothy Maclay[(2)] and J.P. Sheehan[(2)]

[(1)] University of Southampton, Astronautics Research Group, Faculty of Engineering and Physical Sciences, Boldrewood Innovation Campus, Southampton SO16 7QF, UK, hglewis@soton.ac.uk | sdd@soton.ac.uk
[(2)] OneWeb, 1785 Greensboro Station Pl., Tower 3, Suite 400, McLean, VA 22102, USA

ABSTRACT

Due to the significant computational demands involved in the long-term projection of large debris populations, evolutionary models make use of fast and efficient algorithms for collision risk assessment. A commonly used algorithm for collision risk assessment is the cube method, introduced by Liou et al. in 2003. Relatively little research has been undertaken to understand the collision probability estimation errors that arise from the use of the cube method. These errors are especially important when assessing the collision risk associated with large constellations because of the structure and the relatively high number of satellites involved. This paper investigates the cube method, with the aim of determining how the choice of the cube size and time-step affects collision rate estimates. The work used the University of Southampton's Debris Analysis and Monitoring Architecture to the Geosynchronous Environment (DAMAGE) tool to build hypotheses about the influence of the model parameters on the collision rate estimates for orbiting satellites in idealized scenarios, including a large constellation. The results showed that cube sizes between 4 km and 16 km were required to deliver collision rate estimates for discrete object pairs, with smaller cube sizes requiring time-steps of less than 0.1 days. For long-term projections involving many objects in the Low Earth Orbit (LEO) environment, such small time-steps are impractical, as simulation run-times for a single Monte Carlo run would increase from hours to days, or even weeks.

1 INTRODUCTION

Due to the significant computational demands involved in the long-term projection of large debris populations, evolutionary models make use of fast and efficient algorithms for orbit propagation and collision risk assessment. By necessity, these algorithms represent simplifications of the real-world and only provide estimates of true motion and collision probabilities. A commonly used algorithm for collision risk assessment is the cube method, introduced by [1]. A recent investigation of the cube method by [2], focusing on the case of the Jovian moons, has shown that errors in the collision probability estimates for object pairs arise due to an insufficient number of iterations of the method and due to a failure to identify an appropriate size for the cubic volume element. As indicated by [2], these errors are especially important when assessing the collision risk associated with large constellations because of the structure and the relatively high number of satellites involved.

Here, the Debris Analysis and Monitoring Architecture to the Geosynchronous Environment (DAMAGE) evolutionary model was used to build hypotheses about the influence of the key cube model parameters on the collision rate estimates for orbiting satellites. To do this, an idealized scenario was identified where it was possible to determine the true collision rate that would be expected from the cube implementation.

1.1 The Cube Approach

The cube approach was developed as a tool for evaluating collision probabilities between orbiting objects taking into account the needs and constraints of evolutionary models. According to [1], the objective of cube is to estimate the long-term collision probabilities using a “sampling in time” approach. The number of collisions, N_{tot}, occurring between two objects i and j over a long period of time between t_{begin} and t_{end} is,

$$N_{tot} = \int_{t_{begin}}^{t_{end}} P_{i,j}(t)dt, \tag{1}$$

where $P_{i,j}$ is the collision rate. If the projection period is divided into L uniform time intervals $[t_{c+1} - t_c]$ and the collision characteristics are constant over this interval, then

$$N_{tot} = \int_{c=0}^{c=L} [t_{c+1} - t_c]\, P_{i,j}(c)dc. \tag{2}$$

The method assumes that a collision is possible only when two objects are co-located within a small, cubic volume element. Then, the collision rate is,

$$P_{i,j} = s_i s_j V_{imp} \sigma U, \tag{3}$$

where s_i and s_j are the spatial densities of object i and object j in the cube, V_{imp} is the relative velocity, σ is the collision cross-sectional area, and $U = d^3$ is the volume of the cube of length (size) d. Equation 3 is based on the kinetic theory of gas and assumes that objects i and j are equally likely to be found anywhere within the cube. For two objects i and j, both in circular orbits with the same orbital inclination (90°) and semi-major axis, the collision geometry is straightforward (Fig. 1). The orbits intersect twice; once over each of the Earth's poles. If a cube is positioned such that the orbits intersect within it, then the spatial density for each object in the cube is

$$s_i, s_j = \frac{t_{exit} - t_{entry}}{\tau U} = \frac{t_{exit} - t_{entry}}{\tau d^3}, \tag{4}$$

which is proportional to the fraction of the orbit period, τ, spent by each object within the cube. For the specific case of Fig. 1a, the velocity is constant on the circular orbits and Eq. 4 can be written as

$$s_i, s_j = \frac{\theta}{2\pi d^3}, \tag{5}$$

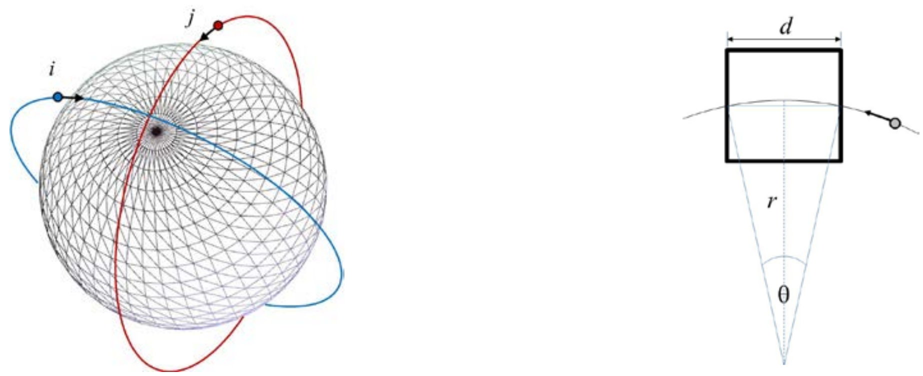
where θ is the central angle of the sector formed by the two orbit radii and the orbit arc connecting the cube entry and exit points (Fig. 1b).

If the cube size is much smaller than the orbit size ($d << r$) then the orbit arc (Fig. 1b) may be assumed to be equal to the cube size, and the central angle is

$$\theta = \frac{d}{r}. \tag{6}$$

Thus the spatial density of each object in the cube volume element is,

$$s_i, s_j = \frac{1}{2\pi r d^2}. \tag{7}$$



(a) (b)

Fig. 1. Two satellites in circular, polar orbits intersecting at 90° above the North and South poles (a), and sector geometry used to estimate the central angle θ for a cube of size d at distance r from the centre of the Earth (b)

Hence, the collision rate estimate from the cube method in Eq. 3 can be written for the orbiting objects i and j in Fig. 1a intersecting in two locations, above the North and South poles, as,

$$P_{i,j} = \frac{V_{imp}\sigma}{2\pi^2 r^2 d}\,. \tag{8}$$

Inspection of Eq. 8 reveals that the collision rate between these two objects is inversely proportional to the cube size, d, when calculated using the cube approach. This dependence emerges for any two orbits, whether they are circular or elliptical. As such, the collision rate will decrease if larger cube sizes are used.

The collision rate is also directly proportional to the combined collision cross-sectional area, σ, which is parameter that can be estimated using a number of different approaches, each resulting in different collision rates. A common approach assumes that objects i and j are spherical in shape (Fig. 2), and σ corresponds to the circular area created by combining the radius of each object [3, 4]:

$$\sigma = \pi\left(R_i + R_j\right)^2. \tag{9}$$

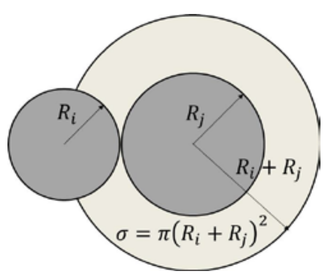


Fig. 2. Collision cross-sectional area for two spherical objects.

If the two objects i and j are identical, such that $R = R_i = R_j$, then Eq. 9 simplifies to

$$\sigma = 4\pi R^2. \tag{10}$$

The collision rate is also directly proportional to the relative velocity, V_{imp}. For the collision geometry in Fig. 1, the relative velocity is

$$V_{imp} = \sqrt{2}\sqrt{\frac{\mu}{r}}, \tag{11}$$

where $\mu = 3.986 \times 10^{14}$ m^3 s^{-2} is the gravitational constant for the Earth. So, after simplification, we have

$$P_{i,j} = \frac{2R^2}{\pi d}\sqrt{\frac{2\mu}{r^5}}, \tag{12}$$

Note that r in Eq. 12 is equivalent to the semi-major axis, a, as the orbits are circular. The characteristics of the true collision nature of the system can be estimated if we assume perfect knowledge of the position of both objects. In this case, the two objects can only collide if the distance between their centers is less than or equal to the sum of their radii and assuming both objects are spherical. If $R \ll r$ then the orbit trajectories close to the intersection can be assumed to be linear and, for the case shown in Fig. 1a, the maximum 'miss-distance' is $2R/\sqrt{2}$. The true collision rate for this case (after simplification) is thus

$$P_{i,j} = \left(\frac{2RN}{\pi r}\right)^2 \ \text{yr}^{-1} \tag{13}$$

where N is the number of orbital revolutions by objects i and j in one year (of τ_y seconds)

$$N = \frac{\tau_y}{2\pi\sqrt{\frac{r^3}{\mu}}} \tag{14}$$

In implementations of the cube method the collision rate is estimated using a combination of sampling in time and a Monte Carlo simulation approach. That is, the position of each object is determined at each time interval (sample) $c = \{0, \ldots, L\}$ and an estimate of $P_{i,j}$ is constructed over many such intervals and many Monte Carlo simulations. Typically, the Mean Anomalies of objects are randomized at each time interval. Then, over a sufficient number of samples and Monte Carlo runs, estimates of the residential probability and, hence, the collision rate can be determined. However, two issues tend to prevent reliable estimates from being determined. Firstly, the time interval $[t_{c+1} - t_c]$ used for most future projections by evolutionary models is typically a period of days, so there are relatively few time intervals within the projection period (i.e. L is low). It is possible to compensate for this by making use of many Monte Carlo simulations, but this becomes computationally expensive and the objects within each simulation can be different. Secondly, the orbits of every object will evolve over time. This means that it is not possible to generate a reliable estimate of $P_{i,j}$ for any given collision geometry.

2 PARAMETRIC STUDY

Two cases were investigated using the DAMAGE code:

- **Case 1**: two satellites ($R = 2.82$ m) in 7000 km radius, circular, polar orbits intersecting at 90° above the North and South poles (Fig. 1).
- **Case 2**: 3814 active and inactive satellites of a large constellation, and 18 selected "target" satellites from the background population.

The constellation case, which was fundamentally the same as reported in [5], was created by simulating the following:

- Walker-delta constellation comprising 1080 satellites in 20 orbital planes at 1100 km altitude and inclined at 85°
- Constellation satellite design lifetime of 5 years, 200 kg and 1 sq. metre
- Constellation build-up from 1 Jan 2018 to 1 Jan 2021 with 20 launches per year and 18 satellites per launch
- Constellation replenishment from 1 Jan 2021 to 1 Jan 2033 with nominally 12 launches per year and 18 satellites

per launch, with failures replaced on-demand. The first replenishment launches commenced on 1 January 2023
- PMD of 90% of constellation spacecraft to a 300 × 1100 km or disposal orbit using electric propulsion
- Deployment from launch vehicle at 400 km followed by low-thrust transfer to 1100 km in 100 days. Immediate de-orbit of rocket bodies following deployment.

The population of 3814 constellation satellites still on-orbit on 1 January 2033 from a single Monte Carlo projection was used as the initial population for case 2. The start time of the simulation was set to 1 January 2013 and proceeded as for case 1. No objects from the background population were included (the 18 target orbits were added at the end). In addition, the assumption of a random Mean Anomaly at each time interval was modified in the following way: the Mean Anomalies of active constellation satellites and failed satellites above 1000 km were propagated, whilst the Mean Anomalies of failed satellites below 1000 km and satellites on disposal orbits were randomized at each time-step.

In each the above cases, the satellites were propagated without orbital perturbations over an 80-year projection period from 1 January 2013 at time-steps selected from the tables below. All conjunctions between the satellites were identified using the cube algorithm and were recorded for subsequent analysis.

For each of these cases, two key simulation parameters were varied: the cube size and the time-step used to compute the evaluation metrics. Table 2 and Table 3 detail the cube sizes and time-steps used in the simulations. The number of conjunctions observed in the simulations was dependent on the choice of parameters. In some cases, no conjunctions were recorded and no collision rate estimates could be obtained. These instances are marked with ○ in the tables below. The number of simulations performed for each parameter setting is shown at the end of each row and column (the first number is the number of simulations where conjunctions were recorded).

Table 1. Cube sizes and time-steps investigated in case 1 (● = conjunctions observed in some/all MC runs, ○ = no conjunctions observed in any MC runs).

	Cube size (km)													**No.**
Time-step (days)	**1**	**3**	**5**	**7.5**	**10**	**12.5**	**15**	**17.5**	**20**	**40**	**60**	**80**	**100**	
0.01	●				●					●	●	●	●	6/6
0.1	○	○	●		●				●	●	●	●	●	7/9
0.5				●	●	●	●	●	●	●	●	●	●	10/10
1	○	○	○	○	●	●	●	●	●	●	●	●	●	9/13
3	○												●	1/2
5	○				●		●		●	●	●	●	●	7/8
10	○												●	1/2
No. simulations	1/6	0/2	1/2	1/2	5/5	2/2	3/3	2/2	4/4	5/5	5/5	5/5	7/7	41/50

Table 2. Cube sizes and time-steps investigated in case 2 (● = conjunctions observed in some/all MC runs).

	Cube size (km)													**No.**
Time-step (days)	**1**	**3**	**5**	**7.5**	**10**	**12.5**	**15**	**17.5**	**20**	**40**	**60**	**80**	**100**	
0.01														0/0
0.1	●		●		●		●		●					5/5
0.5														0/0
1	●		●		●		●		●	●				6/6
3														0/0
5	●		●		●		●		●	●				6/6
10														0/0
No. simulations	3/3	0/0	3/3	0/0	3/3	0/0	3/3	0/0	3/3	2/2	0/0	0/0	0/0	17/17

For the cases listed above, DAMAGE was used to estimate the collision rate, $P_{i,j}$, via a computation based on the cumulative collision probability observed over the 80-year period. The collision rate estimate – the gradient of the line – was determined using the LINEST function in Microsoft Excel.

3 RESULTS

3.1 Case 1 Results

The number of conjunction events detected per year as a function of the cube size was observed to be proportional to the square of the cube size. More than 30 conjunctions were detected per year for a cube size of 100 km but only one conjunction event every 10,000 years for a cube size of 1 km and time-step of 0.01 days. In the latter case, one conjunction event would be detected by DAMAGE every 125 Monte Carlo runs on average for an 80-year projection. The event rate was also inversely proportional to the time-step. An estimate of the true conjunction event rate for a cube size of 100 km would be between 30 and 300 per year, and for a cube size of 10 km would be between 0.1 and 1 per year.

The collision rate estimates generated by DAMAGE for the two satellites in case 1 are shown in Fig. 3 with trend-lines. A subset of the results, for cube sizes < 30 km are shown in Fig. 4. All of the trend-lines followed (approximately) the $1/d$ relationship identified in Eq. 8 with small cube sizes (≤ 10 km) generating relatively high collision rates of 10^{-6} to 10^{-5} per year, and large cube sizes (≥ 20 km) generating relatively low collision rates of 10^{-7} to 10^{-6} per year. The dark grey lines in Fig. 3 and Fig. 4 are based on the estimate of the true collision rate (from Eq. 13) for the two objects in case 1. One can assume that the most reliable estimates of the collision rate come from solutions using short time-steps (as these provide the greatest number of samples and, therefore, the highest number of conjunction events from which to compute the collision rate). The results in Fig. 4 show that cube sizes between 4 km and 16 km are required to deliver collision rate estimates that are close to the true value. The typical cube size that used for DAMAGE simulations, based on the assumptions in [1], is 10 km.

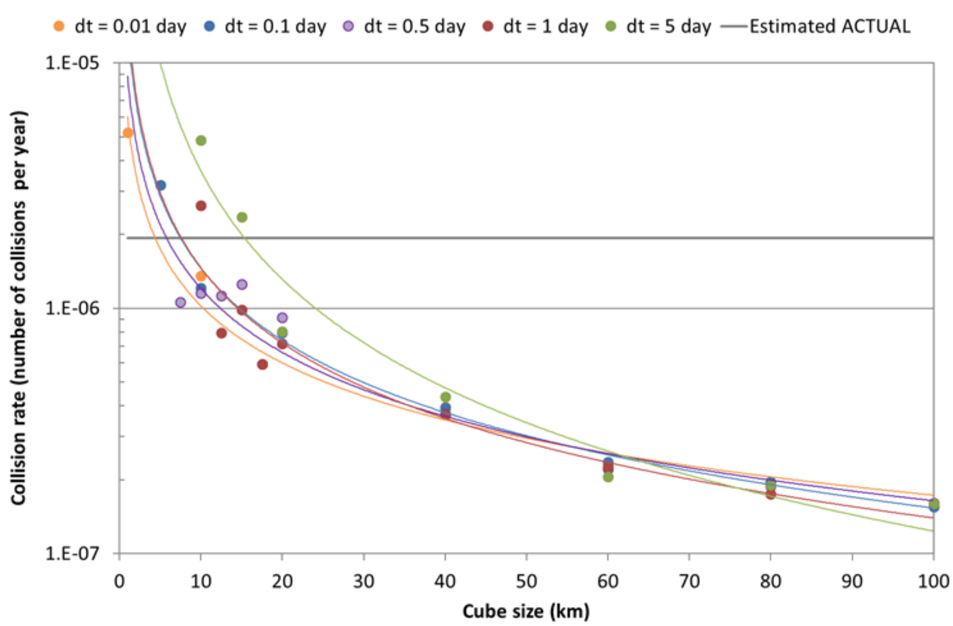


Fig. 3. Collision rate estimates from DAMAGE for case 1 and cube sizes up to 100 km: two objects in circular, polar orbits intersecting at 90° above the North and South poles. The dark grey line is the estimate of the true collision rate determined using Eq. 13.

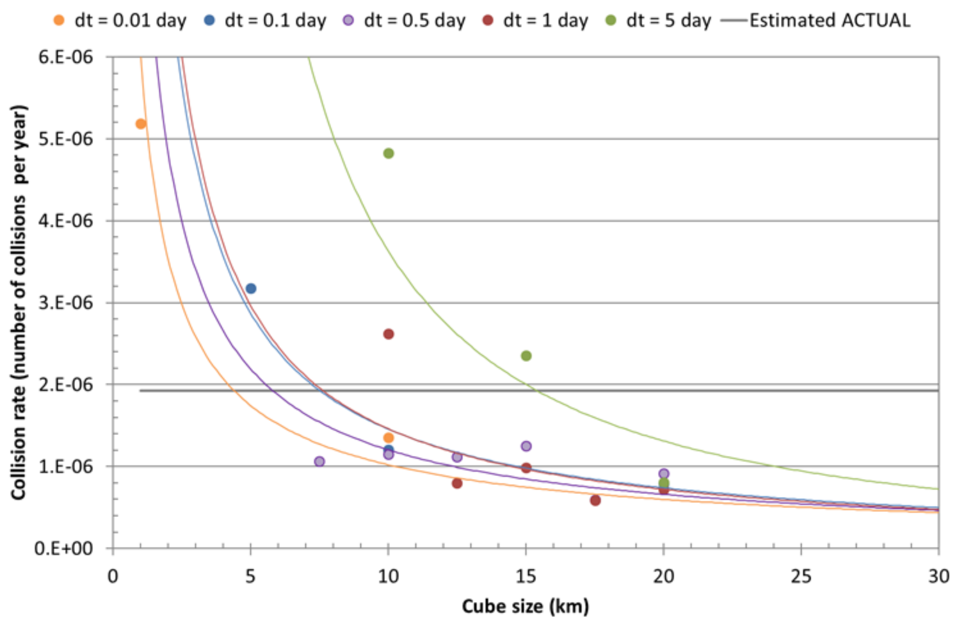


Fig. 4. Collision rate estimates from DAMAGE for case 1 and cube sizes up to 30 km: two objects in circular, polar orbits intersecting at 90° above the North and South poles. The dark grey line is the estimate of the true collision rate determined using Eq. 13.

For relatively large cube sizes, the collision rate estimates were highly correlated ($R^2 > 0.99$) regardless of the time-step used. In contrast, small cube sizes (shown in Fig. 4) often resulted in relatively few conjunctions being recorded except for small time-steps (0.01-0.1 days), and even then the collision rate estimates were poorly correlated. Table 4 shows the proportion of Monte Carlo runs in which at least one conjunction event was detected, with red cells indicating collision rate estimates that were considered to be unreliable. As shown, the typically used cube size of 10 km provided collision rate estimates considered to be reliable only for time-steps of 0.1 days or 0.01 days. For long-term projections involving many objects in the LEO environment, such small time-steps are impractical, as simulation run-times would increase from hours to days, or weeks, for a single Monte Carlo run. For the typically used time-step (5 days) only cube sizes > 20 km were found to detect a sufficient number of conjunction events from which reliable collision rate estimates could be found.

Table 3. Proportion (%) of Monte Carlo runs in which at least one conjunction event was detected by DAMAGE over the 80-year projection. Red cells indicate that the resulting collision rate estimates were considered to be unreliable, whereas green cells indicate that the resulting collision rate estimates were considered to be reliable.

	Cube size (km)												
Time-step (days)	**1**	**3**	**5**	**7.5**	**10**	**12.5**	**15**	**17.5**	**20**	**40**	**60**	**80**	**100**
0.01	1.8				92.9					100	100	100	100
0.1	0	0	6		17				56.5	98.5	100	100	100
0.5				2	4	7	12.5	9.5	21.5	56.5	77	95.5	99.5
1	0	0	0	0	4	2.5	4.5	9.5	7	31	52	72.5	92
3	0												55
5	0				1.5		2.5		2	6.5	13	27	43
10	0												22

3.2 Case 2 results

The conjunction event rate was observed to be a quadratic function of the cube size, with just under 100,000 conjunctions detected per year for a cube size of 20 km but only 37 conjunction events for a cube size of 1 km and time-step of 0.1 days. In addition, the conjunction event rate was inversely proportional to the time-step. Following the approach outlined for case 1, the results suggested that the true conjunction event rate for a cube size of 10 km would be $O(10^7)$ to $O(10^8)$ per year, and for a cube size of 1 km would be between $O(10^4)$ and $O(10^5)$ per year. Given the expectation from case 1 that a cube size of between 4 km and 16 km is needed to estimate the true collision rate, and the assumptions made for the implementation of the cube algorithm, the results suggest that satellites of a large constellation would experience a significant number of conjunction events each year.

The collision rate estimates generated by DAMAGE for the constellation and target satellites in case 2 are shown in Fig. 5. For this case, the overall collision rate was not proportional to $1/d$ due to the increase in satellites found within each cube, as the cube size increased. Observation of the trend-lines in Fig. 5 suggested that the collision rate was proportional to approximately $1/\sqrt{d}$ but the results lacked consistency. For a cube size of 10 km, the collision rate was approximately 0.02 per year; taken with the event rate results above, this suggests a low collision probability per conjunction event, perhaps resulting in relatively few collision avoidance maneuvers.

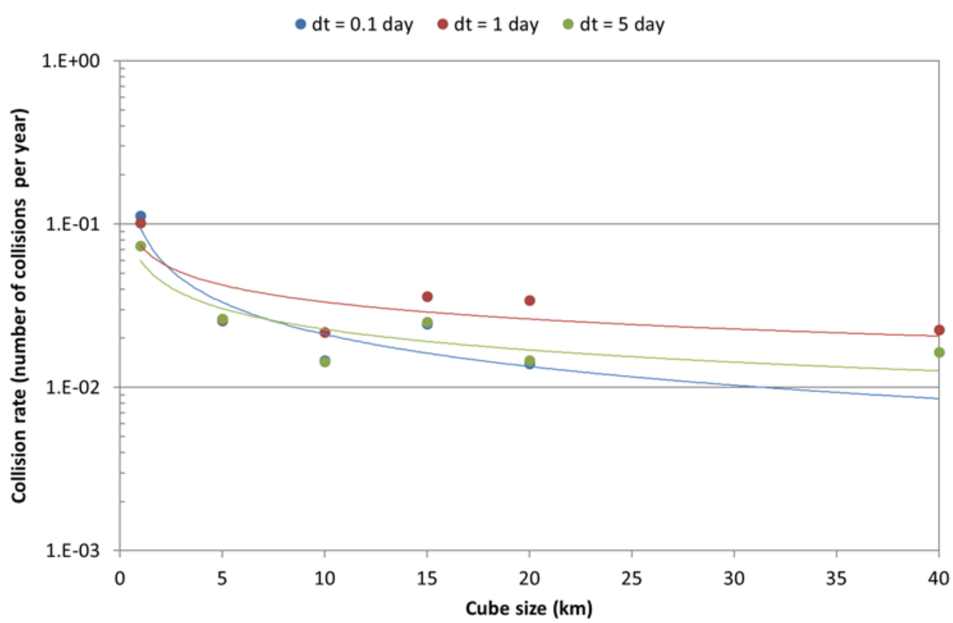


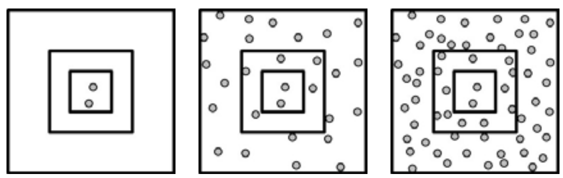
Fig. 5. Collision rate estimates for all satellites from DAMAGE for case 2 and cube sizes up to 40 km: active and inactive constellation satellites.

4 DISCUSSION

The cube method for estimating the collision rates between two objects in intersecting orbits generates collision rates that are inversely proportional to the size of the cube (Eq. 8). By investigating synthetic cases involving only two objects in carefully chosen, ideal orbits (case 1, above), it has been possible to calculate "true" collision rates for these systems and then to identify the cube size that would produce these rates. The results from case 1 indicate that cube sizes between 4 km and 16 km provide good estimates of the true collision rate.

However, the results from case 1 also showed that the time-steps that are typically used for long-term projections of the space debris environment (five days) might not provide sufficient opportunities to sample the system to generate reliable collision rate estimates for some cube sizes. Indeed, the results showed that time-steps of less than 1 day would be required for cube sizes smaller than 10 km in order to detect a sufficient number of conjunctions from which the collision rates could be determined. This would represent a substantial increase in the overall run-time for a typical simulation study. In addition, orbital perturbations cause the collision geometry to change over relatively short time-scales (e.g. important changes can occur after only a few orbits) so it is unlikely that reliable collision rate estimates would be obtained unless many Monte Carlo runs were used.

When considering a larger population (e.g. case 2, which contained 3832 satellites in total) the dependence of the collision rate on the cube size was found to be different, when the overall collision rate was considered (i.e. the cumulative collision rate computed across all satellites in the simulation). Here the collision rate was proportional to approximately$1/\sqrt{d}$. For this case, larger cubes were likely to capture more than two satellites compared with smaller cubes. The two-satellite cases considered above are equivalent to the case shown in Fig. 6a: as the cube size increases the overall spatial density decreases because the number of objects remains unchanged and Eq. 8 applies. In contrast, as the cube size increases in Fig. 6c, the spatial density remains unchanged because the number of unique co-occurring pairs in each cube increases in proportion to the size of the cube. The DAMAGE results observed in Fig. 5 are better explained by the case shown in Fig. 6b, in which the number of unique co-occurring pairs in each cube increases, but in a way that results in a decreasing spatial density, and collision rate, and thus a weaker dependence on the cube size. Nevertheless, the cube size still determines the overall collision rate.



(a) (b) (c)

Fig. 6. Two-dimensional representation of satellites contained in cubes of variable size.

In order to generate collision rates, the cube method makes a fundamental assumption that two objects found within the same cube can collide. In typical long-term projections of the space debris environment where a 10 km cube size is recommended [1] this means that any objects within 17.3 km (the distance along the diagonal of a cube of side 10 km) are assessed for their collision risk. For objects that are randomly distributed throughout near-Earth space (as represented by Fig. 6c) or are subject to random orbital perturbations (e.g. atmospheric drag) and are not under active orbital control this is a reasonable assumption to make; collision rates that are estimated using the cube method with appropriate time-step and cube-size parameters will approximate the true collision rates. However, when the orbits of objects are controlled or are not subject to random orbital perturbations, this fundamental assumption can result in collision rate estimates that are not correct. In the worst case, use of the cube method may result in a prediction of a collision that cannot take place. In the context of the OneWeb satellite constellation, or similar constellations, where many satellites are in orbits that are in close proximity but are tightly controlled, the cube method may therefore be inappropriate.

To gain a better understanding of this issue, a new simulation study was performed. For this investigation, the orbit of object j from case 1 was modified by either increasing or decreasing its semi-major axis by 1 km, 3 km or 8 km. The results showed that the event rate and collision rate estimates were largely consistent regardless of the difference in semi-major axis, yet for all of the cases no collisions were possible because the orbits for objects i and j did not intersect. This result is of particular importance if an assessment is made of a large constellation of satellites where (a) the orbital planes are separated in altitude as a measure to reduce the probability of collisions and (b) the altitude is such that there are no significant perturbations affecting the size or shape of the orbit. If the altitude separation is less than the size of the cube then the cube algorithm may incorrectly identify collisions between satellites – active or otherwise – in neighboring planes. A solution to this issue is to use cube sizes that are smaller than the differences in the altitudes of the orbital planes. However, as the results from case 1 have shown, such cube sizes would require very small time-steps in order to achieve reliable collision rate estimates.

5 CONCLUSIONS

Two simulation cases (and one additional case, described in the section above) have been investigated in a parametric study aimed at identifying appropriate cube-size and time-step combinations for a study of the OneWeb satellite constellation reported in [6]. For the cube method used, the results from these cases have shown that:

- The cube method generates collision rate estimates that are inversely proportional to the size of the cube for a system of two satellites, and approximately inversely proportional to the square root of the size of the cube for a system of many satellites;
- Cube sizes of between 4 km and 16 km are needed to provide good estimates of the true collision rate between two satellites, with time-steps of less than 1 day for cube sizes less than 10 km; and
- The cube method is unable to distinguish between two satellites on orbits that intersect and two satellites that approach to within a few kilometres on orbits that do not intersect.

The combination of these factors means that long-term projections of a large satellite constellation would ideally need small cube sizes and short time-steps if the cube method were used. Without such values, it is likely that the collision rates would not be reliable. The cube algorithm does introduce errors as a result of the kinetic gas theory assumption applied to resident objects, when those resident objects are constrained to orbits that do not intersect by design (such as when constellation planes are separated in altitude and inclination). Therefore, it is recommended that an alternative collision risk method be used for the assessment of constellations, especially when designs include the separation of orbital planes.

6 ACKNOWLEDGEMENTS

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